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Conventional Formulations of Noether's Theorem in Classical Field Theory

JOE ROSEN

Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv, Israel

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Abstract

The role of invariance considerations in conventional formulations of Noether's theorem in classical field theory is investigated and found weaker than is usually supposed. It is shown how nonfulfilment of the conventional assumptions going into Noether's theorem brings about nonconservation.

1. *Introduction*

In recent work (Rosen, 1970) we radically generalized Noether's theorem in classical field theory so as to abolish the role of invariance considerations in the theorem. In this article we investigate conventional formulations of Noether's theorem and especially the assumptions that go into them, for the purpose of clarifying the role of invariance considerations in these formulations. It is shown that to produce a continuity equation (from which a conservation law is obtained†) it is insufficient that an infinitesimal transformation either be a symmetry transformation or leave the action invariant. Both properties must hold. This is not a new result and should be clear, for example, from Hill's (1951) exposition of Noether's theorem. On the other hand, we find, perhaps surprisingly, that even within conventional formulations Noether's theorem allows continuity equations to be associated with transformations that are neither symmetry transformations nor leave the action invariant. We then generalize slightly and obtain a set of assumptions about the transformation, wider than the set of assumptions going into conventional formulations but containing them, and a continuity equation associated with each assumption. Finally we show how nonfulfilment of the conventional assumptions brings about addition of a source term to what otherwise would have been a continuity equation (and therefore nonconservation of what otherwise would have been conserved).

† We do not treat the step leading from continuity equations to conservation laws. See for example Bogoliubov & Shirkov (1959).

2. *Transformations*

Consider a Lagrangian density $\mathscr{L}(x,\varphi(x),d\varphi(x))$ which is a function of the space-time coordinates $x=(x^\mu)$, $\mu=1, 2, 3, 4$, the independent fields $\varphi(x)=(\varphi_i(x))$, $i=1,\ldots, N$, and the first derivatives of the fields $d\varphi(x)$, where $d=(d_\mu)=(d/dx^\mu)$ denotes total derivation with respect to $x=(x^\mu)$.† (All our results are valid, with appropriate generalizations, for $\mathscr{L}$ a function also of higher-order field derivatives up to any finite order. Our restriction here is only for the sake of clarity.) The summation convention holds for space-time indices and for field component indices.

The action functional is

$$J_V[\varphi]=\int_V \mathscr{L}(x,\varphi(x),d\varphi(x))\,d^4x \tag{2.1}$$

where V is the volume of integration.

Consider an arbitrary infinitesimal transformation, where the coordinates transform

$$x\to\bar{x}=x+\delta x(x) \tag{2.2}$$

the fields undergo a variation in their functional form‡

$$\varphi(x)\to\bar{\varphi}(x)=\varphi(x)+\bar{\delta}\varphi(x,\varphi(x),d\varphi(x),\ldots) \tag{2.3}$$

and the Lagrangian density changes its functional form

$$\begin{aligned}\mathscr{L}(x,\varphi(x),d\varphi(x))\to&\ \bar{\mathscr{L}}(x,\varphi(x),d\varphi(x))\\ &=\mathscr{L}(x,\varphi(x),d\varphi(x))+\bar{\delta}\mathscr{L}(x,\varphi(x),d\varphi(x))\end{aligned} \tag{2.4}$$

Write

$$\bar{\delta}\mathscr{L}=d_\mu\bar{\delta}\mathscr{L}_1{}^\mu+\bar{\delta}\mathscr{L}_2 \tag{2.5}$$

where $d_\mu\bar{\delta}\mathscr{L}_1{}^\mu$ contains all divergence terms of $\bar{\delta}\mathscr{L}$. Since the $\bar{\delta}$-variation of φ and $\mathscr{L}$ does not involve a change of x, we have that $\bar{\delta}$ and d commute.

The action transforms as

$$J_V[\varphi]\to\overline{J_V}[\bar{\varphi}]=\int_{\bar{V}}\bar{\mathscr{L}}(\bar{x},\bar{\varphi}(\bar{x}),\overline{d\varphi}(\bar{x}))\,d^4\bar{x} \tag{2.6}$$

We then have to first order

$$\begin{aligned}\delta J&=\bar{J}_{\bar{V}}[\bar{\varphi}]-J_V[\varphi]\\ &=\int_V[\delta\mathscr{L}]\,d^4x\end{aligned} \tag{2.7}$$

where

$$[\delta\mathscr{L}]=d_\mu\left(\mathscr{L}\delta x^\mu+\frac{\partial\mathscr{L}}{\partial d_\mu\varphi_i}\bar{\delta}\varphi_i+\bar{\delta}\mathscr{L}_1{}^\mu\right)+(\bar{\delta}\varphi_i)E^i\mathscr{L}+\bar{\delta}\mathscr{L}_2 \tag{2.8}$$

† For example $d_\mu\varphi(x)=\partial_\mu\varphi(x)$ and $d_\mu f(x,\varphi(x))=[\partial_\mu+(d_\mu\varphi_i)\,\partial/\partial\varphi_i]f(x,\varphi(x))$.
‡ Note that $\delta\varphi=\bar{\varphi}(\bar{x})-\varphi(x)=\bar{\delta}\varphi+(\delta x^\mu)\,d_\mu\varphi$.

and

$$E^i \mathscr{L} = \left(\frac{\partial}{\partial \varphi_i} - d_\mu \frac{\partial}{\partial d_\mu \varphi_i}\right) \mathscr{L} \tag{2.9}$$

E^i being the Euler-Lagrange operator for the field φ_i. The jacobian of transformation (2.2) is $(1 + d_\mu \delta x^\mu)$ and was taken into account in deriving equations (2.7)–(2.9).

The equations of motion are

$$E^i \mathscr{L} \overset{0}{=} 0 \tag{2.10}$$

where $\overset{0}{=}$ denotes equality holding when the fields $\varphi(x)$ satisfy the equations of motion.

3. *Noether's Theorem*

One conventional formulation of Noether's theorem† involves two distinct assumptions:

Assumption A. With respect to a transformation (2.2)–(2.5) the action is invariant for all volumes of integration, i.e., $\delta J = 0$ for all V. From equation (2.7) this implies and is implied by

$$[\delta \mathscr{L}] = 0 \tag{3.1}$$

Equations (2.8) and (3.1) then give

$$d_\mu Z^\mu = -(\bar{\delta}\varphi_i) E^i \mathscr{L} - \bar{\delta}\mathscr{L}_2 \tag{3.2}$$

where

$$Z^\mu = \mathscr{L}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial d_\mu \varphi_i} \bar{\delta}\varphi_i + \bar{\delta}\mathscr{L}_1{}^\mu \tag{3.3}$$

up to a divergenceless vector. This assumption is realized by imposing a condition on $\mathscr{L}$ and its transformation (2.4), (2.5). From equations (2.1), (2.6), (2.7) a necessary and sufficient condition for this is

$$\bar{\mathscr{L}}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4\bar{x} = \mathscr{L}(x, \varphi(x), d\varphi(x))\, d^4 x \tag{3.4}$$

i.e., that $\mathscr{L}$ transform under transformation (2.2)–(2.5) as a scalar density.

Assumption B. With respect to a transformation (2.2)–(2.5) the lagrangian density is form-invariant up to a divergence, i.e.,

$$\bar{\delta}\mathscr{L}_2 = 0 \tag{3.5}$$

† See for example Hill (1951).

from equations (2.4), (2.5). Equations (3.2) and (3.5) then give the continuity equation†

$$\begin{aligned} d_\mu Z^\mu &= -(\bar{\delta}\varphi_i)\, E^i \mathscr{L} \\ &\overset{0}{=} 0 \end{aligned} \tag{3.6}$$

with Z^μ as above. Note that Assumption B is a sufficient, but not necessary, condition for form-invariance of the equations of motion (2.10).‡ This is seen by applying the Euler-Lagrange operator E^i to equations (2.4), (2.5) and using the identity (Courant & Hilbert, 1953)

$$E^i d_\mu \bar{\delta}\mathscr{L}_1{}^\mu = 0 \tag{3.7}$$

Form-invariance of the equations of motion is often taken to characterize symmetry transformations. So if we ignore the nonnecessity of Assumption B for such form-invariance, then Assumption B is that the transformation be a symmetry transformation.§

Thus Assumption A and Assumption B are together sufficient to produce continuity equation (3.6). But they are hardly necessary. An alternate conventional formulation of Noether's theorem starts with a single assumption which is both necessary and sufficient for continuity equation (3.6). Rather than simply state this assumption, we derive it from Assumptions A and B to show its relation to them.

Using the inverse jacobian relation

$$d^4 x = (1 - d_\mu \delta x^\mu)\, d^4 \bar{x} \tag{3.8}$$

rewrite equation (3.4) (Assumption A) as

$$\bar{\mathscr{L}}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4 \bar{x} = \mathscr{L}(x, \varphi(x), d\varphi(x))\, (1 - d_\mu \delta x^\mu)\, d^4 \bar{x} \tag{3.9}$$

Substitute equation (3.5) (Assumption B) in equations (2.4), (2.5), take the transformed coordinates and fields $(\bar{x}, \bar{\varphi}(\bar{x}))$ for the arguments of the functions, and multiply through by $d^4 \bar{x}$ to obtain to first order

$$\begin{aligned} \bar{\mathscr{L}}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4 \bar{x} &= \mathscr{L}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4 \bar{x} + \bar{d}_\mu \bar{\delta}\mathscr{L}_1{}^\mu\, d^4 \bar{x} \\ &= \mathscr{L}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4 \bar{x} \\ &\quad + (d_\mu \bar{\delta}\mathscr{L}_1{}^\mu)(1 - d_\mu \delta x^\mu)\, d^4 \bar{x} \end{aligned} \tag{3.10}$$

† Note that instead of equation (3.5) the weaker condition $\bar{\delta}\mathscr{L}_2 \overset{0}{=} 0$ also gives a continuity equation.

‡ A condition both necessary and sufficient is $E^i \bar{\delta}\mathscr{L}_2 \overset{0}{=} 0$.

§ In previous work (Rosen, 1970) we showed that the transformations leaving the equations of motion form-invariant do not exhaust the set of symmetry transformations. This, together with its nonnecessity for such form-invariance, actually makes Assumption B twice removed from assuming a symmetry transformation.

Since the left-hand sides of equations (3.9) and (3.10) are the same, equate their right-hand sides, cancel out $d^4\bar{x}$, rearrange terms, and get

Assumption C.

$$\mathscr{L}(\bar{x},\bar{\varphi}(\bar{x}),\overline{\mathscr{d}\varphi}(\bar{x})) = [\mathscr{L}(x,\varphi(x),\mathscr{d}\varphi(x)) - \mathscr{d}_\mu \bar{\delta}\mathscr{L}_1{}^\mu](1-\mathscr{d}_\mu \delta x^\mu) \quad (3.11)$$

Continuity equation (3.6) is obtained directly from equation (3.11) by expanding the left-hand side about $\mathscr{L}(x,\varphi(x),\mathscr{d}\varphi(x))$ and using equations (2.2), (2.3). Assumption C is weaker than Assumptions A and B together. $\bar{\delta}\mathscr{L}_2$ does not appear in equation (3.11), so equation (3.5) needs not hold and $\mathscr{L}$ does not have to be form-invariant up to divergence, i.e., transformation (2.2)–(2.5) does not have to be a symmetry transformation. If in fact the transformation is not a symmetry transformation, then $\mathscr{L}$ will not be a *scalar* density, and instead of equation (3.4) we will have

$$\bar{\mathscr{L}}(\bar{x},\bar{\varphi}(\bar{x}),\overline{\mathscr{d}\varphi}(\bar{x}))\,d^4\bar{x} = [\mathscr{L}(x,\varphi(x),\mathscr{d}\varphi(x)) + \bar{\delta}\mathscr{L}_2]\,d^4x \quad (3.12)$$

with resulting noninvariance of the action

$$\delta J = \int_V \bar{\delta}\mathscr{L}_2\,d^4x \quad (3.13)$$

and

$$[\delta\mathscr{L}] = \bar{\delta}\mathscr{L}_2 \quad (3.14)$$

instead of equation (3.1). Equations (2.8) and (3.14) then reproduce continuity equation (3.6), as consistency demands.

Assumption C is not conveniently expressible in terms such as 'scalar density', 'form-invariance', or 'symmetry'. Stated verbally it is that the untransformed Lagrangian density taken as a function of the transformed coordinates and fields be equal to the same function of the untransformed coordinates and fields up to the inverse jacobian factor and a divergence term. Sometimes Assumption C is stated without the inverse jacobian factor.† Then unit jacobian ($\mathscr{d}_\mu \delta x^\mu = 0$) must be assumed, to enable derivation of a continuity equation. This is not valid for, say, dilation transformations.‡ In many field theory texts the divergence term is deleted also.§

So it is clear that even in a conventional formulation of Noether's theorem not only symmetry transformations produce associated continuity equations; any transformation (2.2)–(2.5) obeying Assumption C does this. On the other hand, a transformation's being a symmetry transformation (Assumption B) is not in itself sufficient to produce a continuity equation, but it must also leave the action invariant (Assumption A). And neither is Assumption A sufficient in itself, but must be accompanied by Assumption B.

† For example in Bacry *et al.* (1970).
‡ $\delta x = \rho x$. The jacobian is $(1+4\rho)$.
§ For example in Bjorken & Drell (1965).

4. *Further Considerations*

In this section we yield to that eternal temptation not to leave well enough alone, and we perform a slight generalization. By some additional manipulations it is possible to obtain a wider set of assumptions about transformation (2.2)–(2.5), which includes Assumptions C and $A+B$, and an associated continuity equation for each assumption. To this end define

$$\begin{aligned}\delta\mathscr{L} &= \bar{\mathscr{L}}(\bar{x},\bar{\varphi}(\bar{x}),\overline{d\varphi}(\bar{x})) - \mathscr{L}(x,\varphi(x),d\varphi(x))\\ &= \mathscr{L}(\bar{x},\bar{\varphi}(\bar{x}),\overline{d\varphi}(\bar{x})) - \mathscr{L}(x,\varphi(x),d\varphi(x)) + \bar{\delta}\mathscr{L}\end{aligned} \tag{4.1}$$

Expanding and using equations (2.2)–(2.5), we get

$$\delta\mathscr{L} = (\delta x^\mu)\, d_\mu\mathscr{L} + (\bar{\delta}\varphi_i)\, E^i\,\mathscr{L} + d_\mu\left(\frac{\partial\mathscr{L}}{\partial d_\mu\varphi_i}\bar{\delta}\varphi_i\right) + d_\mu\bar{\delta}\mathscr{L}_1{}^\mu + \bar{\delta}\mathscr{L}_2 \tag{4.2}$$

Manipulate this to obtain

$$\begin{aligned}d_\mu&\left(\mathscr{L}\delta x^\mu + \frac{\partial\mathscr{L}}{\partial d_\mu\varphi_i}\bar{\delta}\varphi_i + \bar{\delta}\mathscr{L}_1{}^\mu\right)\\ &= -(\bar{\delta}\varphi_i)\, E^i\,\mathscr{L} + \delta\mathscr{L} - \bar{\delta}\mathscr{L}_2 + \mathscr{L} d_\mu\delta x^\mu\end{aligned} \tag{4.3}$$

Now add $d_\mu A^\mu$, with A^μ arbitrary, to each side,

$$\begin{aligned}d_\mu&\left(\mathscr{L}\delta x^\mu + \frac{\partial\mathscr{L}}{\partial d_\mu\varphi_i}\bar{\delta}\varphi_i + \bar{\delta}\mathscr{L}_1{}^\mu + A^\mu\right)\\ &= -(\bar{\delta}\varphi_i)\, E^i\,\mathscr{L} + \delta\mathscr{L} - \bar{\delta}\mathscr{L}_2 + \mathscr{L} d_\mu\delta x^\mu + d_\mu A^\mu\end{aligned} \tag{4.4}$$

Then the condition

$$\delta\mathscr{L} - \bar{\delta}\mathscr{L}_2 + \mathscr{L} d_\mu\delta x^\mu + d_\mu A^\mu = 0 \tag{4.5}$$

is sufficient for the continuity equation

$$\begin{aligned}d_\mu W^\mu &= -(\bar{\delta}\varphi_i)\, E^i\,\mathscr{L}\\ &\overset{0}{=} 0\end{aligned} \tag{4.6}$$

to hold, where

$$W^\mu = \mathscr{L}\delta x^\mu + \frac{\partial\mathscr{L}}{\partial d_\mu\varphi_i}\bar{\delta}\varphi_i + \bar{\delta}\mathscr{L}_1{}^\mu + A^\mu \tag{4.7}$$

up to a divergenceless vector. Equation (4.5) for all A^μ forms the set of assumptions mentioned above, and equation (4.6) with equation (4.7) is the continuity equation associated with each assumption.

Assumption C is obtained from equation (4.5) by taking $A^\mu = 0$ and using equations (4.1) and (2.5). Then equations (4.6), (4.7) become continuity equation (3.6). Assumption C with divergence term deleted follows from equation (4.5) by taking $A^\mu = -\bar{\delta}\mathscr{L}_1{}^\mu$. For unit jacobian put $d_\mu\delta x^\mu = 0$.

To obtain Assumptions A (in the form of equation (3.4)) and B from equation (4.5), just take $A^\mu = 0$, as for Assumption C, and put $\bar{\delta}\mathscr{L}_2 = 0$. Use equations (3.8) and (4.1) to see that this is equivalent to equation (3.4).

Another possibility is

$$A^\mu = -\bar{\delta}\mathscr{L}_1{}^\mu - \mathscr{L}\delta x^\mu \tag{4.8}$$

Then equation (4.5) becomes

$$\delta\mathscr{L} - \bar{\delta}\mathscr{L} - (\delta x^\mu)\, d_\mu \mathscr{L} = 0 \tag{4.9}$$

and the associated continuity equation is

$$d_\mu\left(\frac{\partial \mathscr{L}}{\partial d_\mu \varphi_i}\bar{\delta}\varphi_i\right) \overset{0}{=} 0 \tag{4.10}$$

Equation (4.9) expresses invariance of $\mathscr{L}$ under variations in the functional form of $\varphi(x)$, equation (2.3). This is appropriate to local field transformations.

The preceding slightly generalized formulation was developed for the purpose of expressing the set of assumptions, equation (4.5), in terms of $\delta\mathscr{L}$. It is completely equivalent to modifying Assumption C, equation (3.11), by the addition of $-d_\mu A^\mu$ to the right-hand side and obtaining the associated continuity equation as before.

5. *Nonconservation*

We return now to the conventional formulations of Noether's theorem and consider the noncontinuity equations associated with nonfulfilment of the conventional assumptions. Let the nonfulfilment of Assumption C, equation (3.11), be expressed by

$$\mathscr{L}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x})) = [\mathscr{L}(x, \varphi(x), d\varphi(x)) - d_\mu \bar{\delta}\mathscr{L}_1{}^\mu + \Delta](1 - d_\mu \delta x^\mu) \tag{5.1}$$

where Δ contains no divergence terms. Instead of continuity equation (3.6) we then have the divergence equation with source

$$\begin{aligned} d_\mu Z^\mu &= -(\bar{\delta}\varphi_i)\, E^i \mathscr{L} + \Delta \\ &\overset{0}{=} \Delta \end{aligned} \tag{5.2}$$

where Z^μ is given by equation (3.3) up to a divergenceless vector.

Let the nonfulfilment of Assumption A be

$$\delta J = \int_V \Delta_A\, d^4 x \tag{5.3}$$

for all V, where Δ_A does not contain a divergence term, or equivalently

$$[\delta\mathscr{L}] = \Delta_A \tag{5.4}$$

instead of equation (3.1), or equivalently

$$\bar{\mathscr{L}}(\bar{x}, \bar{\varphi}(\bar{x}), \overline{d\varphi}(\bar{x}))\, d^4\bar{x} = [\mathscr{L}(x, \varphi(x), d\varphi(x)) + \Delta_A]\, d^4 x \tag{5.5}$$

instead of equation (3.4). And let the nonfulfilment of Assumption B, equation (3.5), be

$$\bar{\delta}\mathscr{L}_2 = \Delta_B \tag{5.6}$$

where Δ_B contains no divergence term. The resulting divergence equation with source is then

$$\begin{aligned} d_\mu Z^\mu &= -(\bar{\delta}\varphi_i)\, E^i \mathscr{L} + \Delta_A - \Delta_B \\ &\overset{0}{=} \Delta_A - \Delta_B \end{aligned} \tag{5.7}$$

with Z^μ as above. As we saw in Section 3, when the nonfulfilments of Assumptions A and B are correlated through $\Delta_A = \Delta_B$, continuity equation (3.6) is recovered.

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References

Bacry, H., Combe, Ph. and Richard, J. L. (1970). Group-Theoretical Analysis of Elementary Particles in an External Electromagnetic Field. I. The Relativistic Particle in a Constant and Uniform Field, *Nuovo Cimento*, **67A**, 267, Appendix C.

Bjorken, J. D. and Drell, S. D. (1965). *Relativistic Quantum Fields*, Section 11.4. McGraw-Hill, Inc., New York.

Bogoliubov, N. N. and Shirkov, D. V. (1959). *Introduction to the Theory of Quantized Fields*, pp. 21–22. Interscience Publishers, Inc., New York.

Courant, R. and Hilbert, D. (1953). *Methods of Mathematical Physics*, Vol. 1, p. 194. Interscience Publishers, Inc., New York.

Hill, E. L. (1951). Hamilton's Principle and the Conservation Theorems of Mathematical Physics. *Reviews of Modern Physics*, **23**, 253.

Rosen, J. (1970). Noether's Theorem in Classical Field Theory, *Annals of Physics*, to be published.